Computers and Mathematics with Applications 58 (2009) 686–692

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Qualitative properties of some higher order difference equations

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ARTICLE INFO

Article history:
Received 1 January 2009
Received in revised form 14 February 2009
Accepted 20 March 2009

Keywords:
Difference equations
Periodic solutions
Stability
Global asymptotic stability

ABSTRACT

We present sufficient conditions which guarantee that all positive solutions of some higher order rational difference equations are global asymptotically stable. The boundedness of the solutions and the existence of prime period two solutions of such equations are also investigated.

1. Introduction

Rational difference equations (or recursive sequences) is an important class of difference equations where they have many applications in real life situations for example the difference equation

$$x_{n+1} = \frac{a + bx_n}{c + x_n}, \quad n \geq 0,$$

has applications in Optics and Mathematical Biology and is known in the literature as the Riccati difference equation, see Saaty [1]. Also the equation

$$x_{n+1} = \frac{1 + x_n}{x_{n-1}}, \quad n \geq 0,$$

has applications in Geometry (see Leech [2]) and in frieze patterns (see Conway and Coxeter [3]). For some recursive sequences, although their forms (or expressions) look very simple, it is extremely difficult to understand thoroughly the global behaviors of their solutions (see, for instance, [4,5]). For more results related to the dynamics of rational difference equations, we refer to [6–11].

In this paper we deal with the boundedness, the existence of prime period two solutions and the global stability of solutions of the following recursive sequences

$$x_{n+1} = \frac{a x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \dots x_{n-k_i}^{l_i} + b x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \dots x_{n-r_j}^{s_j}}{c x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \dots x_{n-k_i}^{l_i} + d x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \dots x_{n-r_j}^{s_j}}, \quad n \geq 0, \tag{1}$$

and

$$y_{n+1} = \frac{\alpha_0 y_n + \alpha_1 y_{n-1} + \dots + \alpha_t y_{n-t}}{\beta_0 y_n + \beta_1 y_{n-1} + \dots + \beta_t y_{n-t}}, \quad n \geq 0, \tag{2}$$

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doi:10.1016/j.camwa.2009.03.092

where $\alpha_0, \alpha_1, \ldots, \alpha_t, \beta_0, \beta_1, \ldots, \beta_t, a, b, c$ and $d \in (0, \infty)$; $i, j, k_0, k_1, \ldots, k_i, l_0, \ldots, l_i, r_0, \ldots r_j, s_0, \ldots, s_j, t \in \{0, 1, 2, \ldots\}$, $k_u \neq r_v$ for all $u = 0, 1, \ldots, i$ and $v = 0, 1, \ldots, j$ and where $l_0 + l_1 + \cdots + l_i = s_0 + s_1 + \cdots + s_j$ with the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0, y_{-t}, y_{-t+1}, \ldots, y_{-1}$ and $y_0 \in (0, \infty)$ where $k = \max\{k_0, k_1, \ldots, k_i, r_0, r_1, \ldots, r_j\}$.

In what follows we set $p = l_0 + l_1 + \cdots + l_i$.

2. Boundedness and periodicity of solutions

In this section we study the boundedness of the solutions for Eqs. (1) and (2).

Theorem 1. *Every solution of Eq. (1) is bounded and persists.*

Proof. Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of Eq. (1). It follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \frac{a x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + b x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j}}{c x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + d x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j}} \\ &\leq \frac{\max\{a, b\}(x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j})}{\min\{c, d\}(x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j})} = \frac{\max\{a, b\}}{\min\{c, d\}}. \end{aligned}$$

Similarly it is easy to see that

$$x_n \geq \frac{\min\{a, b\}}{\max\{c, d\}}.$$

Thus we get

$$0 < m := \frac{\min\{a, b\}}{\max\{c, d\}} \leq x_n \leq \frac{\max\{a, b\}}{\min\{c, d\}} := M < \infty, \quad \text{for all } n \geq 1.$$

Therefore every solution of Eq. (1) is bounded and persists. Hence the result holds. □

The following results deal with the boundednes of the solutions for Eq. (2).

Lemma 1. *For every positive solution $\{y_n\}_{n=-t}^{\infty}$ of Eq. (2),*

$$d^* \leq y_n \leq D, \quad \text{for all } n \geq 1,$$

where d^* and D are positive constants.

Proof. Let $\{y_n\}_{n=-t}^{\infty}$ be a positive solution of Eq. (2). Then it follows that

$$\begin{aligned} y_{n+1} &= \frac{\alpha_0 y_n}{\beta_0 y_n + \beta_1 y_{n-1} + \cdots + \beta_t y_{n-t}} + \cdots + \frac{\alpha_t y_{n-t}}{\beta_0 y_n + \beta_1 y_{n-1} + \cdots + \beta_t y_{n-t}} \\ &\leq \frac{\alpha_0}{\beta_0} + \frac{\alpha_1}{\beta_1} + \cdots + \frac{\alpha_t}{\beta_t} := D. \end{aligned}$$

Then

$$y_n \leq D \quad \text{for all } n \geq 1.$$

By the change of variables $y_n = \frac{1}{z_n}$ for all $n \geq 1$, Eq. (2) can be rewritten in the form

$$\begin{aligned} z_{n+1} &= \frac{\beta_0 z_{n-1} z_{n-2} \ldots z_{n-t} + \beta_1 z_n z_{n-2} \ldots z_{n-t} + \cdots + \beta_t z_n z_{n-1} \ldots z_{n-t+1}}{\alpha_0 z_{n-1} z_{n-2} \ldots z_{n-t} + \alpha_1 z_n z_{n-2} \ldots z_{n-t} + \cdots + \alpha_t z_n z_{n-1} \ldots z_{n-t+1}} \\ &\leq \frac{\beta_0}{\alpha_0} + \frac{\beta_1}{\alpha_1} + \cdots + \frac{\beta_t}{\alpha_t} := \widetilde{d}. \end{aligned}$$

That is

$$y_n \geq \frac{1}{\widetilde{d}} := d^* \quad \text{for all } n \geq 1,$$

and this completes the proof. □

Lemma 2. *Let $\{y_n\}_{n=-t}^{\infty}$ be a positive solution of Eq. (2). Then for some $i,\ j = 0, \ldots, t$,*

$$\frac{\alpha_i}{\alpha_i \widetilde{d} - \beta_i} \leq y_n \leq \frac{\beta_j D - \alpha_j}{\beta_j} \quad \text{for all } n \geq 1,$$

where $\widetilde{d}$ and D are as in Lemma 1.

Proof. Let $G : (0, \infty)^{t+1} \to (0, \infty)$ be a function defined by

$$G(z_0, z_1, \dots, z_t) = \frac{\alpha_0 z_0 + \alpha_1 z_1 + \cdots + \alpha_t z_t}{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_t z_t}.$$

Then for any values of the quotients $\frac{\alpha_0}{\beta_0}, \frac{\alpha_1}{\beta_1}, \dots$, and $\frac{\alpha_t}{\beta_t}$ the function G is non-decreasing in at least one of its arguments and is non-increasing in at least another one of its arguments. Without loss of generality we assume that G is non-decreasing in its argument z_i and is non-increasing in its argument z_j for some $i,\ j = 0, 1, \dots, t$ with $i \neq j$. Then it follows from Eq. (2) that

$$y_{n+1} \leq \frac{\alpha_0 y_n + \cdots + \alpha_{j-1} y_{n-j+1} + \alpha_{j+1} y_{n-j-1} + \cdots + \alpha_t y_{n-t}}{\beta_0 y_n + \cdots + \beta_{j-1} y_{n-j+1} + \beta_{j-1} y_{n-j-1} + \cdots + \beta_t y_{n-t}},$$

and

$$y_{n+1} \geq \frac{\alpha_0 y_n + \cdots + \alpha_{i-1} y_{n-i+1} + \alpha_{i+1} y_{n-i-1} + \cdots + \alpha_t y_{n-t}}{\beta_0 y_n + \cdots + \beta_{i-1} y_{n-i+1} + \beta_{i-1} y_{n-i-1} + \cdots + \beta_t y_{n-t}} \quad \text{for all } n \geq 0.$$

The rest of the proof is similar to that of Lemma 1 and will be left to the reader. □

Theorem 2. *Every solution of Eq. (2) is bounded and persists.*

Proof. Let $\{y_n\}_{n=-t}^{\infty}$ be a positive solution of Eq. (2). As in the proof of Theorem 1, it is easy to see for all $n \geq 1$ that

$$0 < M_1 := \frac{\min\{\alpha_0, \alpha_1, \dots, \alpha_t\}}{\max\{\beta_0, \beta_1, \dots, \beta_t\}} \leq y_n \leq \frac{\max\{\alpha_0, \alpha_1, \dots, \alpha_t\}}{\min\{\beta_0, \beta_1, \dots, \beta_t\}} := M_2 < \infty.$$

Thus the proof is complete. □

Now we study the existence of prime period two solutions of Eq. (1). Set $K = \{k_0, k_1, \dots, k_i\}$ and $R = \{r_0, r_1, \dots, r_j\}$.

Theorem 3. *Assume that* $\Gamma = \sum_{\substack{k_\theta \in K \\ k_\theta\text{-even}}} l_\theta = 1 + \sum_{\substack{r_\psi \in R \\ r_\psi\text{-even}}} s_\psi$. *Then Eq. (1) has positive prime period two solutions if and only if*

$$4da < (c - d)(b - a). \tag{3}$$

Proof. First suppose that there exists a prime period two solution

$$\dots, \eta, \zeta, \eta, \zeta, \dots$$

of Eq. (1). We will prove that (3) holds. We see from Eq. (1) that

$$\eta = \frac{a\eta^{p-\Gamma}\zeta^{\Gamma} + b\eta^{p-\Gamma+1}\zeta^{\Gamma-1}}{c\eta^{p-\Gamma}\zeta^{\Gamma} + d\eta^{p-\Gamma+1}\zeta^{\Gamma-1}} = \frac{a\zeta + b\eta}{c\zeta + d\eta},$$

and

$$\zeta = \frac{a\zeta^{p-\Gamma}\eta^{\Gamma} + b\zeta^{p-\Gamma+1}\eta^{\Gamma-1}}{c\zeta^{p-\Gamma}\eta^{\Gamma} + d\zeta^{p-\Gamma+1}\eta^{\Gamma-1}} = \frac{a\eta + b\zeta}{c\eta + d\zeta}.$$

Now, by some simple computations, it is easy to see that η and ζ are the two positive distinct roots of the quadratic equation

$$t^2 - \frac{(b - a)}{d} t + \frac{a(b - a)}{d(c - d)} = 0, \tag{4}$$

and this makes Inequality (3) true.

Second suppose that Inequality (3) holds and assume that

$$x_{-T} = x_{-T+2} = \cdots = x_{-3} = x_{-1} = \eta$$

and

$$x_{-T+1} = x_{-T+3} = \cdots = x_{-2} = x_0 = \zeta,$$

where $T = \max\{k_i, r_j\}$, T is odd (the case where T is even, is similar), η and ζ are the roots of Eq. (4). Then by direct substitution into Eq. (1) and using Mathematical Induction we obtain that Eq. (1) has a prime period two solution of the form

$$\dots, \eta, \zeta, \eta, \zeta, \dots.$$

The proof is complete. □

Theorem 4. *Eq. (2) has positive prime period two solutions if and only if*

$$4\left(\sum_{\omega=0}^{[\frac{t}{2}]}\alpha_{2\omega}\right)\left(\sum_{\delta=0}^{[\frac{t-1}{2}]}\beta_{2\delta+1}\right) < \left(\sum_{\omega=0}^{[\frac{t-1}{2}]}\alpha_{2\omega+1} - \sum_{\omega=0}^{[\frac{t}{2}]}\alpha_{2\omega}\right)\left(\sum_{\delta=0}^{[\frac{t}{2}]}\beta_{2\delta} - \sum_{\delta=0}^{[\frac{t-1}{2}]}\beta_{2\delta+1}\right),$$

where $[.]$ *denotes the greatest integer function.*

Proof. The proof is similar to the proof of Theorem 3 and so will be omitted. □

3. Stability analysis

In this section we study the global asymptotic stability of Eqs. (1) and (2).

Observe that each of Eqs. (1) and (2) has a unique positive equilibrium point given by $\bar{x} = \frac{a+b}{c+d}$ and $\bar{y} = \frac{\alpha_0+\alpha_1+\cdots+\alpha_t}{\beta_0+\beta_1+\cdots+\beta_t}$ respectively. Therefore the linearized equation of Eq. (1) about the equilibrium $\bar{x}$ is the linear difference equation

$$w_{n+1} - \sum_{u=0}^{i} l_u \frac{(ad-bc)}{(a+b)(c+d)} w_{n-k_u} + \sum_{v=0}^{j} s_v \frac{(ad-bc)}{(a+b)(c+d)} w_{n-r_v} = 0,$$

whose characteristic equation is

$$\phi^{k+1} - \sum_{u=0}^{i} l_u \frac{(ad-bc)}{(a+b)(c+d)} \phi^{k-k_u} + \sum_{v=0}^{j} s_v \frac{(ad-bc)}{(a+b)(c+d)} \phi^{k-k_v} = 0. \tag{5}$$

Theorem 5. *Assume that*

$$2p\,|(ad-bc)| < (a+b)(c+d). \tag{6}$$

Then the positive equilibrium point $\bar{x}$ *of Eq. (1) is global asymptotically stable if*

$$c^{2p-2}\,\bar{x}^{p-1}\,(b^p + c\,d^{p-1}\,\bar{x}^p) > (c+d)\,d^{p-2}\,[a^{2p-1} - (c\bar{x})^{2p-1}]. \tag{7}$$

Proof. It is well known that all roots of Eq. (5) lie in the unit disk if

$$\left|\frac{(ad-bc)}{(a+b)(c+d)}\right| \sum_{u=0}^{i} l_u + \left|\frac{(ad-bc)}{(a+b)(c+d)}\right| \sum_{v=0}^{j} s_v < 1,$$

which is true by (6). Then the equilibrium point of Eq. (1) is locally asymptotically stable. Thus it suffices to show that the equilibrium $\bar{x}$ of Eq. (1) is the global attractor of the solutions of Eq. (1).

Now assume that $ad \geq bc$ (the case $ad \leq bc$ is similar), it follows from Eq. (1) that

$$\begin{aligned} x_{n+1} &= \frac{a x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + b x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j}}{c x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + d x_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j}} \\ &\leq \frac{a x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + b(0)}{c x_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + d(0)} = \frac{a}{c}, \quad n \geq 0. \end{aligned} \tag{8}$$

Similarly

$$x_n \geq \frac{b}{d}, \quad \text{for all } n \geq 1. \tag{9}$$

Then we see from Eqs. (8) and (9) that

$$\frac{b}{d} \leq x_n \leq \frac{a}{c} \quad \text{for all } n \geq 1.$$

Therefore there exist positive constants I and S such that

$$\frac{b}{d} \leq I = \lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n = S \leq \frac{a}{c}.$$

Now we rewrite Eq. (1) in the following form

$$\begin{aligned} x_{n+1} &= F(x_{n-k_0}, x_{n-k_1}, \ldots, x_{n-k_i}, x_{n-r_0}, x_{n-r_1}, \ldots, x_{n-r_j}) \\ &= \frac{ax_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + bx_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j}}{cx_{n-k_0}^{l_0} x_{n-k_1}^{l_1} \ldots x_{n-k_i}^{l_i} + dx_{n-r_0}^{s_0} x_{n-r_1}^{s_1} \ldots x_{n-r_j}^{s_j}}. \end{aligned}$$

Observe that the function $F(w_0, w_1, \ldots, w_i, z_0, z_1, \ldots, z_j)$ is non-decreasing in $w_0, w_1, \ldots, w_i$ and is non-increasing in $z_0, z_1, \ldots, z_j$. Then it follows from Eq. (1) that $x_{n+1} \geq F(I-\epsilon, I-\epsilon, \ldots, I-\epsilon, S-\epsilon, \ldots, S-\epsilon)$ for some $\epsilon > 0$, and so

$$I \geq F(I-\epsilon, I-\epsilon, \ldots, I-\epsilon, S-\epsilon, \ldots, S-\epsilon).$$

Let $\epsilon \to 0$, then $I \geq F(I, I, \ldots, I, S, \ldots, S)$. That is;

$$I \geq \frac{aI^p + bS^p}{cI^p + dS^p}.$$

Then we obtain

$$aI^{2p-1} + bI^{p-1}S^p - cI^{2p} \leq dI^pS^p. \tag{10}$$

Similarly it is easy to see from Eq. (1) that

$$aS^{2p-1} + bI^pS^{p-1} - cS^{2p} \geq dI^pS^p. \tag{11}$$

Therefore it follows from Eqs. (10) and (11) that

$$bI^{p-1}S^{p-1}(I-S) - a\left(I^{2p-1} - S^{2p-1}\right) + c\left(I^{2p} - S^{2p}\right) \geq 0,$$

or

$$(I-S)\left(bI^{p-1}S^{p-1} + cS^{2p-1}\right) - (a-cI)\left(I^{2p-1} - S^{2p-1}\right) \geq 0,$$

or equivalently

$$(I-S)\left[bI^{p-1}S^{p-1} + cS^{2p-1} - (a-cI)\left(I^{2p-2} + SI^{2p-3} + S^2I^{2p-4} + \cdots + S^{2p-3}I + S^{2p-2}\right)\right] \geq 0,$$

and so $I \geq S$ if

$$bI^{p-1}S^{p-1} + cS^{2p-1} - (a-cI)\left(I^{2p-2} + SI^{2p-3} + S^2I^{2p-4} + \cdots + S^{2p-3}I + S^{2p-2}\right) \geq 0. \tag{12}$$

Now it follows from (7) that

$$\begin{aligned} & b^p + c\, d^{p-1}\, \bar{x}^p > \frac{c(c+d)\, d^{p-2}\, \bar{x}^p \left(a^{2p-1} - (c\bar{x})^{2p-1}\right)}{c^{2p-1}\, \bar{x}^{2p-1}}, \\ \Leftrightarrow\quad & \frac{b^p + c\, d^{p-1}\, \bar{x}^p}{d^{p-1}} > \frac{c\, \bar{x}^p\, (ad-bc)\left[\left(\frac{a}{c\,\bar{x}}\right)^{2p-1} - 1\right]}{d\,(a - c\,\bar{x})}, \\ \Leftrightarrow\quad & b\, \bar{x}^{p-1}\left(\frac{b}{d}\right)^{p-1} + c\, \bar{x}^{2p-1} > \frac{c\, \bar{x}^{2p-1}\, (ad-bc)\left[\left(\frac{a}{c\,\bar{x}}\right)^{2p-1} - 1\right]}{d(a - c\,\bar{x})}. \end{aligned}$$

Now

$$b\bar{x}^{p-1}\left(\frac{b}{d}\right)^{p-1} + c\bar{x}^{2p-1} \leq bI^{p-1}S^{p-1} + cS^{2p-1}, \tag{13}$$

and

$$\begin{aligned} \frac{c\bar{x}^{2p-1}\, (ad-bc)\left[\left(\frac{a}{c\,\bar{x}}\right)^{2p-1} - 1\right]}{d(a - c\,\bar{x})} &= \left(a - \frac{bc}{d}\right)\left[\bar{x}^{2p-2} + \left(\frac{a}{c}\right)\bar{x}^{2p-3} + \left(\frac{a}{c}\right)^2\bar{x}^{2p-4} + \cdots + \left(\frac{a}{c}\right)^{2p-3}\bar{x} + \left(\frac{a}{c}\right)^{2p-2}\right] \\ &> (a-cI)\left(I^{2p-2} + SI^{2p-3} + S^2I^{2p-4} + \cdots + S^{2p-3}I + S^{2p-2}\right). \end{aligned} \tag{14}$$

Then we see from (13) and (14) that

$$bI^{p-1}S^{p-1} + cS^{2p-1} > (a-cI)\left(I^{2p-2} + SI^{2p-3} + S^2I^{2p-4} + \cdots + S^{2p-3}I + S^{2p-2}\right),$$

which yields that (12) is satisfied and then the proof is complete. □

In the following we discuss the stability of Eq. (2).

Theorem 6. *Assume that*

$$\sum_{\mu,\ \nu=0}^{t} \left|\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu\right| < \left(\sum_{\zeta=0}^{t}\alpha_\zeta\right)\left(\sum_{\varsigma=0}^{t}\beta_\varsigma\right). \tag{15}$$

Then the positive equilibrium point $\bar{y}$ of Eq. (2) is global asymptotically stable if

$$\alpha_\tau\beta_\sigma \geq \alpha\beta \quad \text{for some } \tau, \sigma = 0, 1, \dots, t \text{ with } \tau \neq \sigma \tag{16}$$

where $\alpha = \sum_{\substack{\mu=0\\ \mu\neq\tau}}^{t} \alpha_\mu$ and $\beta = \sum_{\substack{\nu=0\\ \nu\neq\sigma}}^{t} \beta_\nu$.

Proof. The linearized equation of Eq. (2) about the equilibrium $\bar{y}$ is the linear difference equation

$$z_{n+1} - \sum_{\mu,\ \nu=0}^{t} \frac{(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)}{(\alpha_0+\alpha_1+\cdots+\alpha_t)(\beta_0+\beta_1+\cdots+\beta_t)} z_{n-\mu} = 0,$$

whose characteristic equation is

$$\lambda^{t+1} - \sum_{\mu,\ \nu=0}^{t} \frac{(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)}{(\alpha_0+\alpha_1+\cdots+\alpha_t)(\beta_0+\beta_1+\cdots+\beta_t)} \lambda^{t-\mu} = 0. \tag{17}$$

It is well known that all roots of Eq. (17) lie in the unit disk if

$$\sum_{\mu,\ \nu=0}^{t} \left| \frac{(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)}{(\alpha_0+\alpha_1+\cdots+\alpha_t)(\beta_0+\beta_1+\cdots+\beta_t)} \right| < 1,$$

which is true by (15). Then the equilibrium point of Eq. (2) is locally asymptotically stable.

In the following we prove that $\bar{y}$ is a global attractor of the solutions of Eq. (2).

Now we obtain from Theorem 2 that there exist positive constants h and H such that

$$h = \lim_{n\to\infty} \inf y_n \leq \lim_{n\to\infty} \sup y_n = H.$$

In the following we will prove that $h \geq H$. Again, as in the proof of Theorem 5, we obtain from Eq. (2) that

$$\frac{\alpha h + \alpha_\sigma H}{\beta H + \beta_\tau h} \leq y_{n+1} \leq \frac{\alpha H + \alpha_\sigma h}{\beta h + \beta_\tau H}$$

for some $\tau, \sigma = 0, \dots t$. Thus

$$(\alpha h + \alpha_\sigma H)(\beta h + \beta_\tau H) \leq (\alpha H + \alpha_\sigma h)(\beta H + \beta_\tau h),$$

or

$$(h - H)(h + H)(\alpha_\sigma\beta_\tau - \alpha\beta) \geq 0.$$

Then it follows from (16) that $h \geq H$ which gives that

$$\lim_{n\to\infty} y_n = \bar{y},$$

and so the proof is complete. □

Remark. The obtained results in this work can be applied for the following general difference equation

$$x_{n+1} = \frac{\sum_{\rho=0}^{t} \alpha_\rho x_{n-k_{\rho 0}}^{l_{\rho 0}} x_{n-k_{\rho 1}}^{l_{\rho 1}} \dots x_{n-k_{\rho i}}^{l_{\rho i}}}{\sum_{\rho=0}^{t} \beta_\rho x_{n-k_{\rho 0}}^{l_{\rho 0}} x_{n-k_{\rho 1}}^{l_{\rho 1}} \dots x_{n-k_{\rho i}}^{l_{\rho i}}}.$$

4. Applications

In this section we give some examples as direct applications of the obtained results in this paper.

Example 1. Consider the equation

$$x_{n+1} = \frac{x_n^2 + 5x_{n-1}x_{n-4}}{7x_n^2 + 3x_{n-1}x_{n-4}}. \tag{18}$$

This equation is a special case of Eq. (1) with $k_0 = k_1 = 0$, $l_0 = l_1 = 1$, $l_2 = \cdots = l_i = 0$, $r_0 = 1$, $r_1 = 4$, $s_0 = 1$, $s_1 = 1$ and $s_2 = \cdots = s_j = 0$. Eq. (18) satisfies all hypotheses of Theorem 3 with $\Gamma = 2$, then Eq. (18) has positive prime period two solutions. In fact Eq. (18) has the following solution:

$$\left\{\ldots \frac{1}{3},\ 1,\ \frac{1}{3}, 1,\ \ldots\right\}.$$

Example 2. Consider the equation

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-1}}{\beta_0 x_n + \beta_1 x_{n-1}}. \tag{19}$$

This equation is a special case of Eq. (2) with $t = 1$. Assume that $\alpha_1\beta_0 \geq \alpha_0\beta_1$. Then we get the following:

(I) The condition in Theorem 4 becomes

$$\alpha_1\beta_0 > 3\alpha_0\beta_1 + \alpha_1\beta_1 + \alpha_0\beta_0. \tag{20}$$

(II) The hypotheses of Theorem 6 become

$$\alpha_0\beta_1 \leq \alpha_1\beta_0 < 3\alpha_0\beta_1 + \alpha_1\beta_1 + \alpha_0\beta_0. \tag{21}$$

Then Eq. (19) has prime period two solutions if and only if (20) holds and its positive solutions are globally asymptotically stable if (21) is satisfied. These results are consistent with that obtained in [12].

Example 3. Consider the equation

$$x_{n+1} = \frac{\alpha_0 x_n + \alpha_1 x_{n-1} + \alpha_2 x_{n-2}}{\beta_0 x_n + \beta_1 x_{n-1} + \beta_2 x_{n-2}}. \tag{22}$$

This equation is a special case of Eq. (2) with $t = 2$. Then the condition in Theorem 4 becomes

$$4\beta_1(\alpha_0 + \alpha_1) < (\alpha_1 - \alpha_0 - \alpha)(\beta_0 + \beta_2 - \beta_1). \tag{23}$$

Thus according to Theorem 4, Eq. (22) has prime period two solutions if and only if (23) holds and this result agrees with that given in [6].

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